Grade 7/8 Math Circles
November 20/21/22/23, 2023
Parabolas

Introduction

If you toss a snowball or kick a football in the air, the paths that these objects follow are similar. The curved paths are created by the force of gravity acting upon the object and they are symmetric on either side of its peak like in the image below.

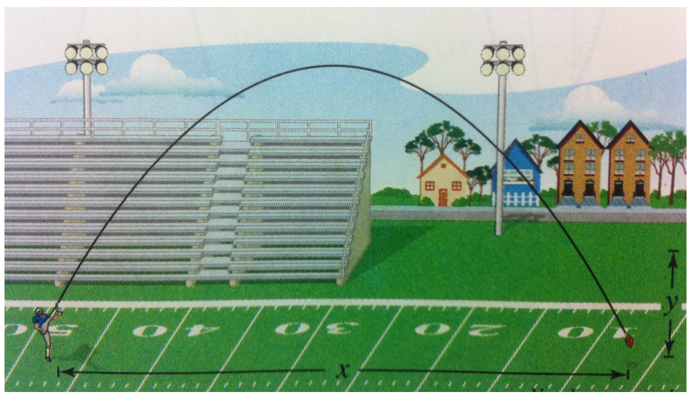


Image retrieved from Alzar School

Satellites also use this curve because of the desirable way in which waves reflect off the dish. More information about the reflection of waves will be seen further in the lesson.

Image retrieved from Wikipedia

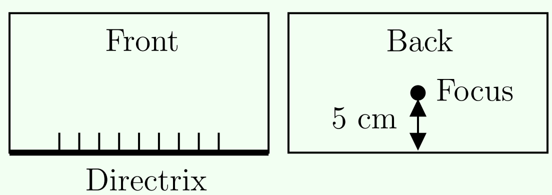
Focus-Directrix Paper Folding Activity

Activity

You need a blank sheet of paper, a pencil, and a ruler for this activity.

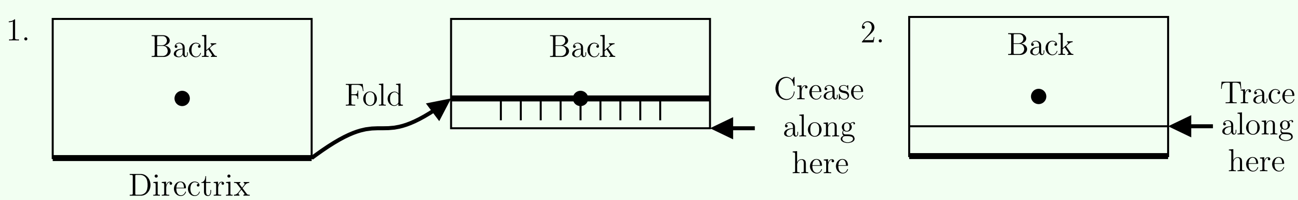
Setup:

1. With your paper in a landscape (horizontal) orientation, center nine evenly spaced tick marks along the bottom edge of the page as shown below. From now on, this edge with the ticks will be referred to as the **directrix**.
2. On the back of the page, make a dot that is 5 cm above the center tick. From now on, this dot will be referred to as the **focus**.
3. Keep your paper with the back side facing up.



Action:

1. Fold the center tick so it touches the focus and crease the page.
2. Lightly trace along this crease with a pencil and a ruler.
3. Repeat the above two steps for each tick. (TIP: The folding is easier if you start from the middle tick and work your way to either side).



This video gives a demonstration of this paper folding activity. https://www.youtube.com/watch?v=sdurzQ4rz9k

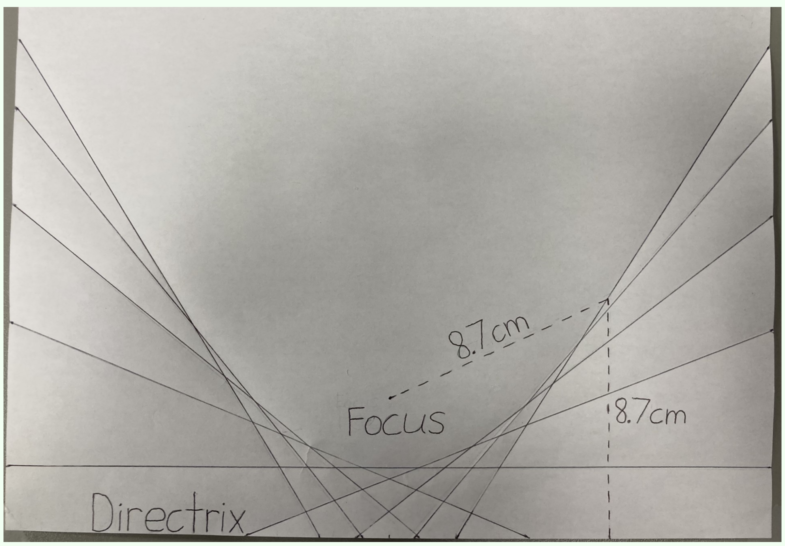
And this is an online version of the paper folding activity. https://www.geogebra.org/m/pxa9zgv8

Activity Continued

After you have folded all nine ticks onto the focus and traced each crease, you should see a curve that resembles the football path and the satellite dish.

Analysis:
Measure from the rightmost tick straight up to the uppermost crease. Then measure from this point on the crease to the focus.
What do you notice about these measurements?

Solution:



This image should look similar to your sheet after completing the activity and your two measurements should be equal. The fact that these measurements are equal actually *defines* the formation of this curve, called a **parabola**!

This lesson is all about the wonders of these curved paths called **parabolas**. You will see various definitions and applications of the parabola. The previous activity outlined the Focus-Directrix definition of the parabola, also called the Locus definition.

Focus-Directrix (or Locus) Definition:

A parabola can be defined as all the points that are an equal distance from a fixed line (directrix) to a fixed point (focus).

Exercise 1

Answer the following questions with this activity https://www.geogebra.org/m/fw9u6zRG.

(a) If you increase the distance between the directrix and the focus, what happens to the parabola's shape?

(b) What about if you decrease the distance between the directrix and the focus?

Function Definition

The parabola can also be defined with algebra and graphing, but first let's review the Cartesian-plane.

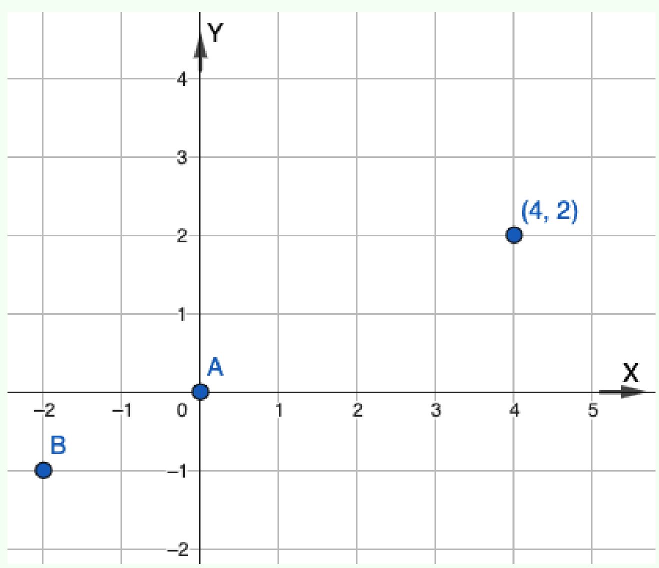
The Cartesian-Plane

The Cartesian-plane is a grid consisting of two axes. Traditionally, the horizontal axis is the x-axis and the vertical axis is the y-axis. Given a set of coordinates, we can plot a point on the Cartesian-plane. Coordinates are always in the form (x, y), meaning the first number is the value on the x-axis and the second number is the value on the y-axis.

Example 1

The Cartesian-plane has the point $(4, 2)$. This means the x value is 4 and the y value is 2. So, we find the number 4 on the x-axis and then move up to the number 2 on the y-axis. This is where the point $(4, 2)$ goes.

Label points A and B using Cartesian coordinates.



Solution:

A $= (0, 0)$ which is a special point called the **origin** and B $= (-2, -1)$.

Functions

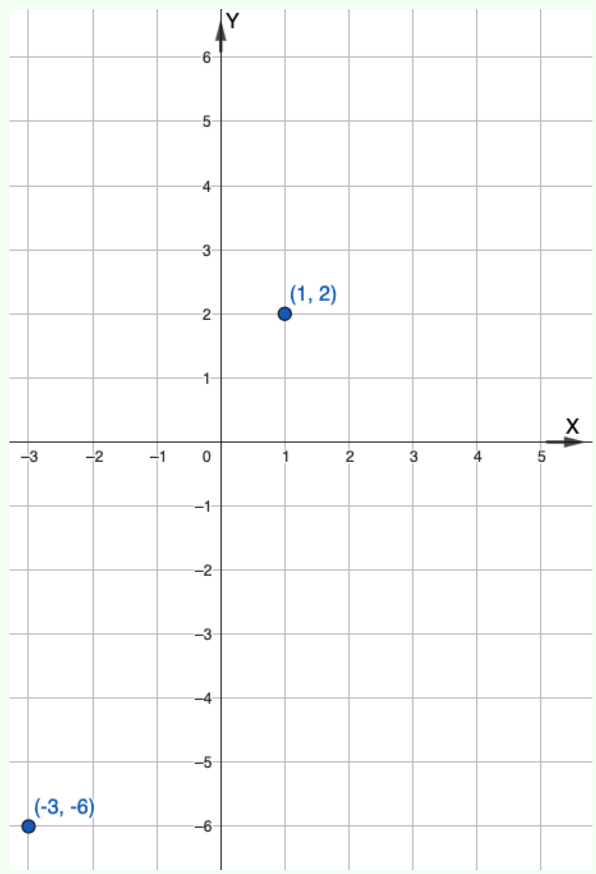
You can think of a **function** as a number machine that takes in a number, does an operation to that number, and then spits out exactly one new number. For example, if the function $y = 2x$ is given $x = 1$ as input, it will multiply 1 by 2 and return $y = 2$. It can do this to any real number. Another example is $x = -3$. It will spit out $y = -6$.

Example 2

To plot the function $y = 2x$, fill in the missing information in the table and plot the coordinates from the table on the graph. Two are done for you.

Then connect the points. What is the shape of this graph?

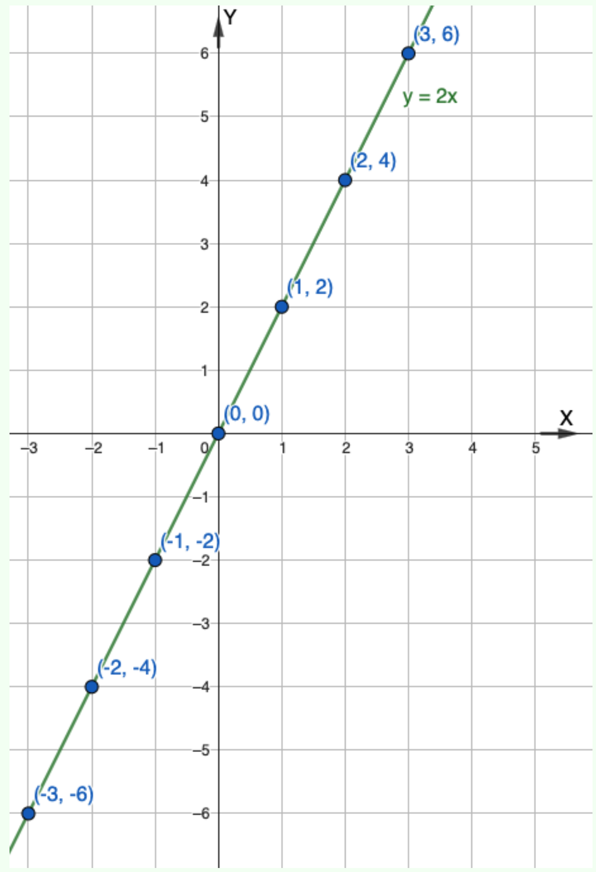
x	$y = 2x$	Coordinate (x, y)
-3	-6	$(-3, -6)$
-2		
-1		
0		
1	2	$(1, 2)$
2		
3		



Example 2 Continued

Solution:

x	$y = 2x$	Coordinate (x, y)
-3	-6	$(-3, -6)$
-2	-4	$(-2, -4)$
-1	-2	$(-1, -2)$
0	-0	$(0, 0)$
1	2	$(1, 2)$
2	4	$(2, 4)$
3	6	$(3, 6)$



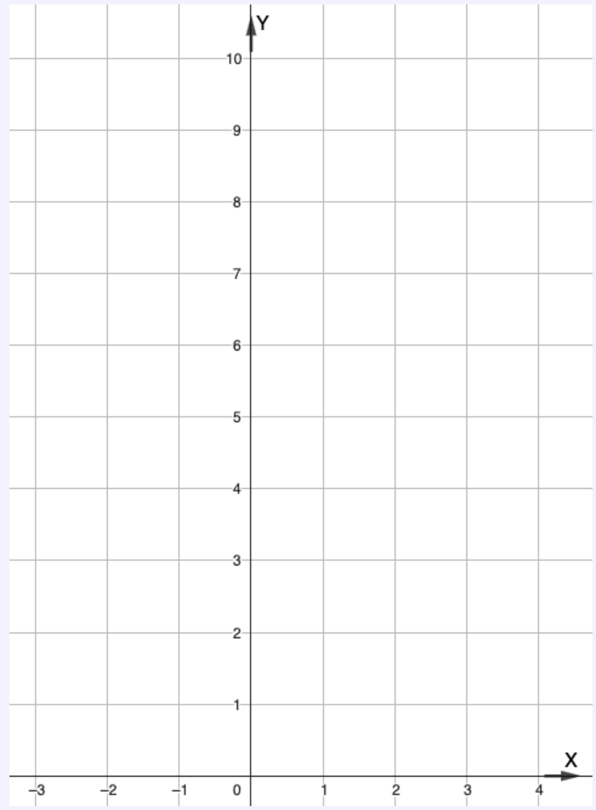
This green line is the graph of the function $y = 2x$. It extends to the left and right forever!

To be a function, the relation must pass the **vertical line test**. The relation passes the vertical line test if at any x-value, a vertical line crosses the graph of the relation *exactly once*. This test guarantees that for every x-value, there is only one y-value. This is the definition of a function. Check to make sure the last example passes this test.

Exercise 2

Graph the function $y = x^2 + 1$ using the table and graph below.

x	$y = x^2 + 1$	Coordinate (x, y)
-3		
-2		
-1		
0		
1		
2		
3		



The last exercise produces another parabolic curve! It is a specific example of a parabola; the one with equation $y = x^2 + 1$. This leads to another definition of the parabola.

Function Definition

A parabola can be defined as the function $y = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.

The **vertex** of a parabola is the point at which the parabola "turns around." In Exercise 2, the vertex is the point $(0, 1)$.

Stop and Think

In the definition above, why must we include $a \neq 0$?

Since a, b, and c, can be *any* real number (except $a \neq 0$), there are infinitely many different parabolas! To explore how their shape can vary, explore this slider activity. Parabola slider: https://www.geogebra.org/m/wvpjqrry.

Exercise 3

Use the slider from above (linked again here; https://www.geogebra.org/m/wvpjqrry) to answer the following questions.

(a) When does the parabola open up. When does it open down?

(b) How does sliding a affect the parabola? How is this related to changing the distance between the focus and directrix?

(c) How does sliding c affect the parabola?

The Area Under a Parabola

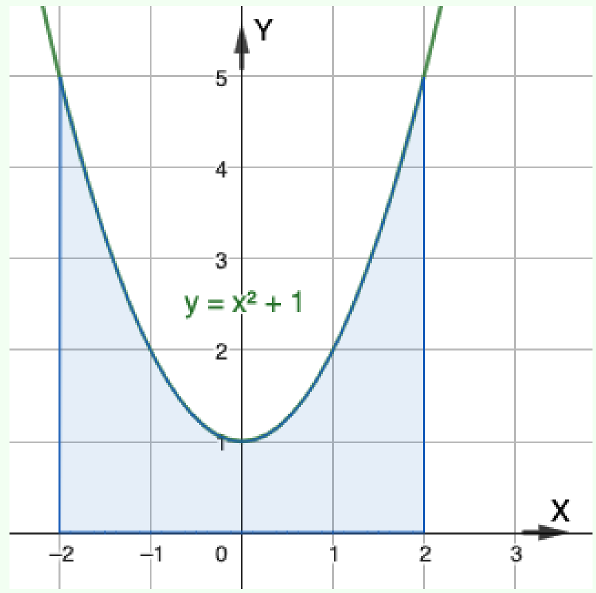
Recall that area is the space that a two-dimensional object occupies. In grade 7 and 8, you learn to calculate the area of polygons (2D shapes with straight sides), but calculating the area of curved shapes is usually left out because this often requires *calculus*. However, nearly 2000 years before the invention of calculus, a Greek mathematician and physicist by the name of Archimedes discovered a method to find the area under a parabola! What is "the area under a parabola?" The next example will help explain this.

Archimedes (287 BC)

Image retrieved from Biography

Example 3

When we say "area under a parabola", we need to specify the bounds on the x-axis. In this example, we are going to calculate the area of the blue region, which is the area under the parabola and above the x-axis from $x = -2$ to $x = 2$.

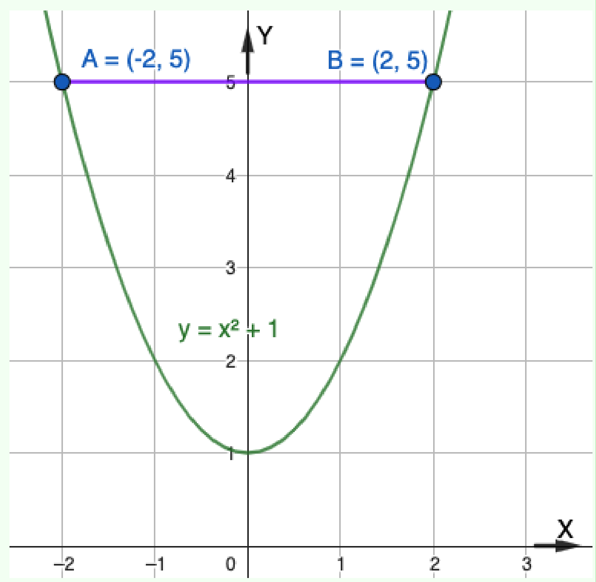


Archimedes' method follows a step-by-step process to calculate the area under a parabola. Follow the steps below to complete this example.

1. Start with the lower bound. Find the y-value when x is the lower bound and plot point A here. Do the same for the upper bound but label this point B. Connect points A and B with a line (use a ruler). This will be called line AB.

   Solution:

   The lower bound is $x = -2$. When $x = -2$, $y = 5$, so point A is $(-2, 5)$. The upper bound is $x = 2$. When $x = 2$, $y = 5$, so point B is $(2, 5)$.



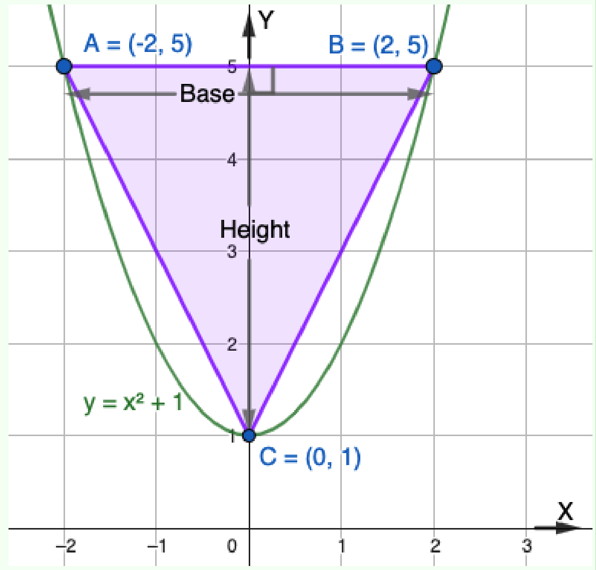
2. Find the area of triangle ABC by labelling point C using the following steps:
   (i) The x-coordinate of point C is always the midpoint of the lower and upper bounds. $x = \frac{\text{lower+upper}}{2}$.
   (ii) Once you have the x-coordinate, find the y-coordinate by plugging x into the function and label this C.

   Solution:

   The x-coordinate of point C is $\frac{-2+2}{2} = 0$.

   At $x = 0$, $y = (0)^2 + 1 = 1$ so C is $(0, 1)$.

   Base = Height = 4 so $A_{ABC} = \dfrac{4 \times 4}{2} = 8$.



3. Archimedes proved that the area enclosed by the line AB and the parabola (the red area) is four-thirds of the area of triangle ABC. Calculate $\frac{4}{3}$ of the area of triangle ABC.

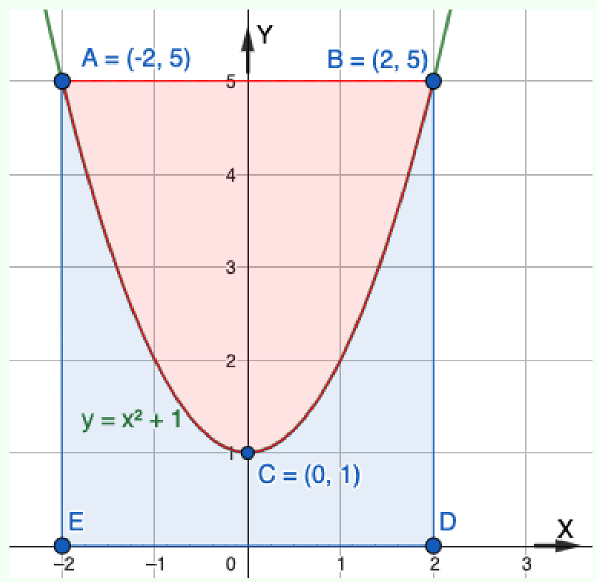
   Solution: $A_{red} = \frac{4}{3} \times 8 = \frac{32}{3}$

4. Finally, calculate the blue area by subtracting the red area from quadrilateral ABDE.

   $A_{blue} = (4 \times 5) - \frac{32}{3}$

   $A_{blue} = \frac{60}{3} - \frac{32}{3}$

   $A_{blue} = \frac{28}{3}$



The above example can also be solved by evaluating $\int_{-2}^{2}(x^2+1)\,dx$. This is an *integral* which calculates the area between functions, but we were able to find the area without integral calculus!

Applications of Parabolas

The start of this lesson mentioned some applications of parabolas. This section will explore these in more detail.

The Football Problem

When someone throws a football, the arc that it creates is a parabola. Since we know the equation of a parabola, it's possible to solve for values such as when/where the ball hits the ground and when/where the ball peaks. Solving for these values requires mathematical skills that are beyond this lesson, but we can still determine these values by looking at the parabola on a graph.

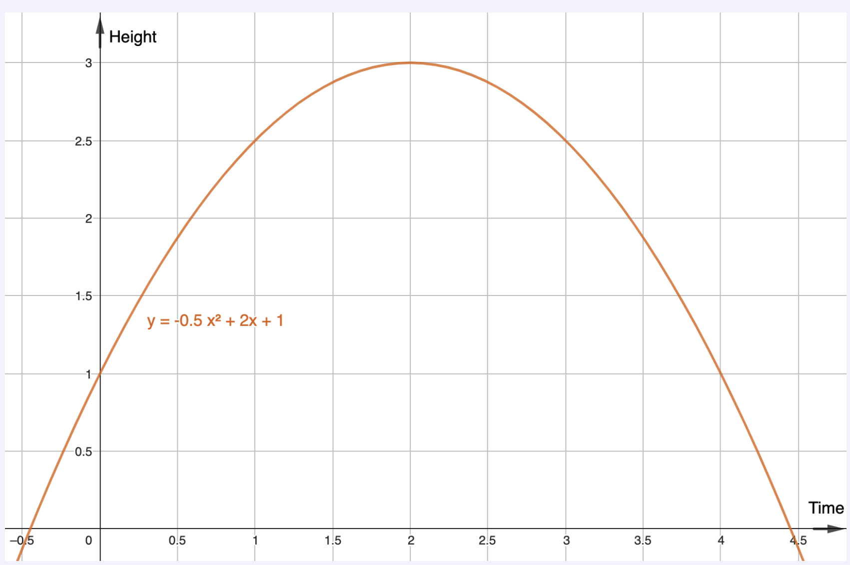
Exercise 4

Suppose the path of a football can be modelled by the function $h = -0.5t^2 + 2t + 1$. Using the graph of this function, answer the questions below. Note that the horizontal axis is the time in seconds and the vertical axis is the height of the ball in metres.

(a) What is the height of the ball at time $t = 0$? Think of a reason the ball starts at this height.

(b) Determine at what time the ball reaches its maximum height, and determine the ball's maximum height.

(c) Determine when the ball hits the ground.



Satellites

Satellites use parabolic curves because of the desirable way in which signals bounce off of the dish. All signals that enter the satellite dish are reflected to the same point called the focus. Wait, we've used that word already today. The focus of a parabolic satellite is the same as the focus in the paper folding activity!

This property is great because now just a signal receiver is placed at the focus to collect *all* incoming signals. Check out this activity to explore this parabolic property! https://www.geogebra.org/m/pUCDYC97

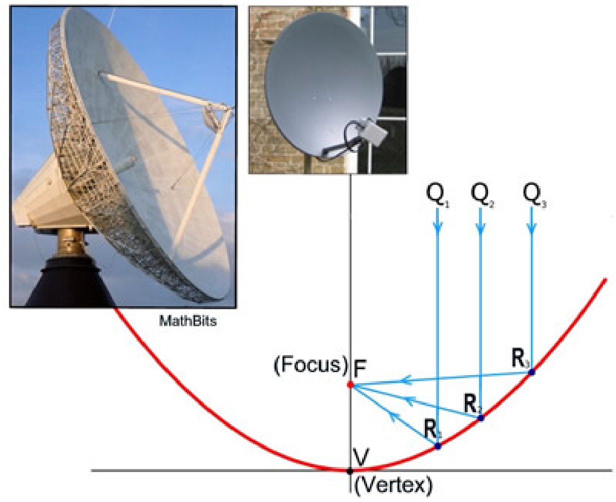


Image retrieved from Math Bits Note Book

Stop and Think

A satellite dish isn't the only example of a parabolic object. Think of some other devices that use the parabola!

Summary

Definitions of the Parabola

- Focus-Directrix Definition: A parabola is the collection of points that are an equal distance between a point and a line.
- Function Definition: The function $y = ax^2 + bx + c$ is the equation of a parabola.